Supplemental Material: Causal Entropic Forces

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PARTICLE IN A BOX DETAILS

The causal entropic force parameters used in the simulation were $\tau = 10$ s, $T_{\mathrm{r}} = 4 \times 10^5$ K, and $T_{\mathrm{c}} = 5\ T_{\mathrm{r}}$. The time step used was $\epsilon = 0.025$ s.

The T_{c} values in this example and the others were chosen such that the observed behaviors were exhibited over approximately 500 time steps. Since the role of T_{c} is to scale the magnitude of the causal entropic force, the primary result of smaller T_{c} values is slower behavior.

The degrees of freedom are summarized in Table I.

TABLE I. Degrees of freedom for particle in a box.

D.O.F. (j)	Forced?	Mass (m_j)	q_j^{min}	q_j^{max}	$q_j(0)$	$p_j(0)$
x	Yes	m	0	L	$L/10$	0
y	Yes	m	0	$L/5$	$L/10$	0

The stochastic equations of motion for the evolution of path microstates (which were sampled to calculate the causal entropic forces) were

$$\dot{p}_x(t) = g_x(\mathbf{x}(t), t) = -p_x(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_x(\lfloor t/\epsilon \rfloor \epsilon) + h_x(\mathbf{x}(t)) \tag{1}$$

$$\dot{p}_y(t) = g_y(\mathbf{x}(t), t) = -p_y(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_y(\lfloor t/\epsilon \rfloor \epsilon) + h_y(\mathbf{x}(t)) \tag{2}$$

$$\dot{q}_x(t) = p_x(t)/m \tag{3}$$

$$\dot{q}_y(t) = p_y(t)/m, \tag{4}$$

and the deterministic equations of motion for the evolution of the causal macrostate (shown in Figure 2(a) and Supplemental Movie 1) when subjected to the combined causal entropic force and expectation energetic force contributions were

$$\dot{p}_x(t) = \langle g_x(\mathbf{x}(t), t) \rangle + F_x(t) = -p_x(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + F_x(t) + h_x(\mathbf{x}(t)) \tag{5}$$

$$\dot{p}_y(t) = \langle g_y(\mathbf{x}(t), t) \rangle + F_y(t) = -p_y(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + F_y(t) + h_y(\mathbf{x}(t)) \tag{6}$$

$$\dot{q}_x(t) = p_x(t)/m \tag{7}$$

$$\dot{q}_y(t) = p_y(t)/m, \tag{8}$$

where $g_j(\mathbf{x}(t), t)$ represent the energetic force components defined in the main text, $f_j(t)$ represent the random force components defined in the main text, $F_j(t)$ represent the causal entropic force components defined in the main text, x, y represent the forced degrees of freedom of the particle, and $h_j(\mathbf{x}(t))$ represent any instantaneous perfectly elastic collision forces of the particle with hard walls of the universe at $q_x = 0, L$ and $q_y = 0, L/5$. The particle mass was $m = 10^{-21}$ kg. The allowed region in system phase space was a rectangle with width $L = 400$ m and height $L/5$, as shown in Table I, with momentum components bounded by $|p_j(t)| \le m_j(q_j^{\max} - q_j^{\min})/\epsilon$.

CART AND POLE DETAILS

The causal entropic force parameters used in the simulation were $\tau = 25$ s, $T_{\mathrm{r}} = 4.0 \times 10^6$ K, and $T_{\mathrm{c}} = 20\, T_{\mathrm{r}}$. The time step used was $\epsilon = 0.05$ s.

The degrees of freedom are summarized in Table II.

TABLE II. Degrees of freedom for cart and pole.

D.O.F. (j)	Forced?	Mass (m_j)	$q_j^{\min}$	$q_j^{\max}$	$q_j(0)$	$p_j(0)$
x	Yes	m	0	L	$L/10$	0
θ	No	Ml^2	0	2π	π	0

The stochastic equations of motion for the evolution of path microstates (which were sampled to calculate the causal entropic forces) were

$$\frac{\dot{p}_x(t)}{m} = \frac{g_x(\mathbf{x}(t), t) + ml\dot{q}_\theta^2(t)\sin q_\theta(t) - mg\sin q_\theta(t)\cos q_\theta(t)}{M + m + m\cos^2 q_\theta(t)} \tag{9}$$

$$= \frac{-p_x(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_x(\lfloor t/\epsilon \rfloor \epsilon) + h_x(\mathbf{x}(t)) + ml\dot{q}_\theta^2(t)\sin q_\theta(t) - mg\sin q_\theta(t)\cos q_\theta(t)}{M + m + m\cos^2 q_\theta(t)} \tag{10}$$

$$\frac{\dot{p}_\theta(t)}{Ml^2} = \frac{\dot{p}_x(t)\cos q_\theta(t)/m + g\sin q_\theta(t)}{l} \tag{11}$$

$$\dot{q}_x(t) = p_x(t)/m \tag{12}$$

$$\dot{q}_\theta(t) = p_\theta(t)/(Ml^2), \tag{13}$$

and the deterministic equations of motion for the evolution of the causal macrostate (shown in Figure 2(b) and Supplemental Movie 2) when subjected to the combined causal entropic force and

expectation energetic force contributions were

$$\frac{\dot{p}_x(t)}{m} = \frac{\langle g_x(\mathbf{x}(t),t)\rangle + F_x(t) + ml\dot{q}_\theta^2(t)\sin q_\theta(t) - mg\sin q_\theta(t)\cos q_\theta(t)}{M+m+m\cos^2 q_\theta(t)} \tag{14}$$

$$= \frac{-p_x(\lfloor t/\epsilon \rfloor \epsilon) + F_x(t) + h_x(\mathbf{x}(t)) + ml\dot{q}_\theta^2(t)\sin q_\theta(t) - mg\sin q_\theta(t)\cos q_\theta(t)}{M+m+m\cos^2 q_\theta(t)} \tag{15}$$

$$\frac{\dot{p}_\theta(t)}{Ml^2} = \frac{\dot{p}_x(t)\cos q_\theta(t)/m + g\sin q_\theta(t)}{l} \tag{16}$$

$$\dot{q}_x(t) = p_x(t)/m \tag{17}$$

$$\dot{q}_\theta(t) = p_\theta(t)/(Ml^2), \tag{18}$$

where $g_j(\mathbf{x}(t),t)$ represent the energetic force components defined in the main text, $f_j(t)$ represent the random force components defined in the main text, $F_j(t)$ represent the causal entropic force components defined in the main text, x is the forced horizontal degree of freedom of the cart, θ is the unforced angular degree of freedom measured from the vertical of the massless pole, $h_x(\mathbf{x}(t))$ represents any instantaneous cart-wall collision forces, $m = 10^{-21}$ kg is the mass at the end of the pole, $M = 10^{-21}$ kg is the mass of the cart, $g = 9.8$ m/s^2 is the gravitational acceleration, and $l = 40$ m is the pole length. The allowed region in system phase space had width $L = 400$ m, as shown in Table II, with momentum components bounded by $|p_j(t)| \le m_j(q_j^{\max} - q_j^{\min})/\epsilon$. Collisions of the cart with the walls at $q_x = 0, L$ were perfectly inelastic, and the pole was allowed full angular freedom.

TOOL USE PUZZLE DETAILS

The causal entropic force parameters used in the simulation were $\tau = 10$ s, $T_\mathrm{r} = 8.0 \times 10^5$ K, and $T_\mathrm{c} = 1.25\ T_\mathrm{r}$. The time step used was $\epsilon = 0.05$ s.

The degrees of freedom are summarized in Table III.

The tube had length 100 m and diameter 80 m. Disc I (with forced degrees of freedom x_1, y_1) had mass $m_1 = m = 10^{-21}$ kg and radius 50 m, Disc II (with unforced degrees of freedom x_2, y_2) had mass $m_2 = 0.5m$ and radius 20 m, and Disc III (with unforced degrees of freedom x_3, y_3) had mass $m_3 = 2m$ and radius 20 m. The allowed region in system phase space was a square of width $L = 400$ m, as shown in Table III, with momentum components bounded by $|p_j(t)| \le m_j(q_j^{\max} - q_j^{\min})/\epsilon$. Disc-disc, disc-wall, and disc-tube collisions were all perfectly elastic.

The stochastic equations of motion for the evolution of path microstates (which were sampled

TABLE III. Degrees of freedom for tool use puzzle.

D.O.F. (j)	Forced?	Mass (m_j)	$q_j^{\min}$	$q_j^{\max}$	$q_j(0)$	$p_j(0)$
x_1	Yes	m	0	L	$0.5L$	0
y_1	Yes	m	0	L	$0.375L$	0
x_2	No	$0.5m$	0	L	$0.5L$	0
y_2	No	$0.5m$	0	L	$0.625L$	0
x_3	No	$2m$	0	L	$0.9L$	0
y_3	No	$2m$	0	L	$0.5L$	0

to calculate the causal entropic forces) were

$$\dot{p}_{x_1}(t) = g_{x_1}(\mathbf{x}(t), t) = -p_{x_1}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_{x_1}(\lfloor t/\epsilon \rfloor \epsilon) + h_{x_1}(\mathbf{x}(t)) \tag{19}$$

$$\dot{p}_{y_1}(t) = g_{y_1}(\mathbf{x}(t), t) = -p_{y_1}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_{y_1}(\lfloor t/\epsilon \rfloor \epsilon) + h_{y_1}(\mathbf{x}(t)) \tag{20}$$

$$\dot{p}_{x_2}(t) = h_{x_2}(\mathbf{x}(t)) \tag{21}$$

$$\dot{p}_{y_2}(t) = h_{y_2}(\mathbf{x}(t)) \tag{22}$$

$$\dot{p}_{x_3}(t) = h_{x_3}(\mathbf{x}(t)) \tag{23}$$

$$\dot{p}_{y_3}(t) = h_{y_3}(\mathbf{x}(t)) \tag{24}$$

$$\dot{q}_{x_1}(t) = p_{x_1}(t)/m \tag{25}$$

$$\dot{q}_{y_1}(t) = p_{y_1}(t)/m \tag{26}$$

$$\dot{q}_{x_2}(t) = p_{x_2}(t)/(0.5m) \tag{27}$$

$$\dot{q}_{y_2}(t) = p_{y_2}(t)/(0.5m) \tag{28}$$

$$\dot{q}_{x_3}(t) = p_{x_3}(t)/(2m) \tag{29}$$

$$\dot{q}_{y_3}(t) = p_{y_3}(t)/(2m), \tag{30}$$

and the deterministic equations of motion for the evolution of the causal macrostate (shown in Figure 3 and Supplemental Movie 3) when subjected to the combined causal entropic force and

expectation energetic force contributions were

$$\dot{p}_{x_1}(t) = \langle g_{x_1}(\mathbf{x}(t), t)\rangle + F_{x_1}(t) \tag{31}$$

$$= -p_{x_1}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + F_{x_1}(t) + h_{x_1}(\mathbf{x}(t)) \tag{32}$$

$$\dot{p}_{y_1}(t) = \langle g_{y_1}(\mathbf{x}(t), t)\rangle + F_{y_1}(t) \tag{33}$$

$$= -p_{y_1}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + F_{y_1}(t) + h_{y_1}(\mathbf{x}(t)) \tag{34}$$

$$\dot{p}_{x_2}(t) = h_{x_2}(\mathbf{x}(t)) \tag{35}$$

$$\dot{p}_{y_2}(t) = h_{y_2}(\mathbf{x}(t)) \tag{36}$$

$$\dot{p}_{x_3}(t) = h_{x_3}(\mathbf{x}(t)) \tag{37}$$

$$\dot{p}_{y_3}(t) = h_{y_3}(\mathbf{x}(t)) \tag{38}$$

$$\dot{q}_{x_1}(t) = p_{x_1}(t)/m \tag{39}$$

$$\dot{q}_{y_1}(t) = p_{y_1}(t)/m \tag{40}$$

$$\dot{q}_{x_2}(t) = p_{x_2}(t)/(0.5m) \tag{41}$$

$$\dot{q}_{y_2}(t) = p_{y_2}(t)/(0.5m) \tag{42}$$

$$\dot{q}_{x_3}(t) = p_{x_3}(t)/(2m) \tag{43}$$

$$\dot{q}_{y_3}(t) = p_{y_3}(t)/(2m), \tag{44}$$

where $g_j(\mathbf{x}(t), t)$ represent the energetic force components defined in the main text, $f_j(t)$ represent the random force components defined in the main text, $F_j(t)$ represent the causal entropic force components defined in the main text, and $h_j(\mathbf{x}(t))$ represent any instantaneous disc-disc, disc-wall, and disc-tube collision forces.

As control experiments – performed implicitly as part of the Monte Carlo path integral calculations – the model was stochastically evolved 400 times from its initial configuration for duration τ. In only 15 out of 400 runs (3.75%) was such random evolution able to release Disc III from the tube such that $|q_{x_3}(\tau) - q_{x_3}(0)| \geq 0.125L$ or $|q_{y_3}(\tau) - q_{y_3}(0)| \geq 0.1L$.

SOCIAL COOPERATION PUZZLE DETAILS

The causal entropic force parameters used in the simulation were $\tau = 20$ s, $T_r = 8.0 \times 10^5$ K, and $T_c = 2.5$ T_r. The time step used was $\epsilon = 0.1$ s.

The degrees of freedom are summarized in Table IV.

TABLE IV. Degrees of freedom for cooperation puzzle.

D.O.F. (j)	Forced?	Mass (m_j)	$q_j^{\min}$	$q_j^{\max}$	$q_j(0)$	$p_j(0)$
x_1	Yes	m	0	$0.5L$	$0.35L$	0
y_1	Yes	m	$0.5L$	L	$0.75L$	0
x_2	Yes	m	$0.5L$	L	$0.65L$	0
y_2	Yes	m	$0.5L$	L	$0.85L$	0
x_3	No	m	$0.12L$	$0.88L$	$0.5L$	0
y_3	No	m	0	L	$0.25L$	0
y_4	No	$0.1m$	0	L	$0.5L$	0

Disc I (with forced degrees of freedom x_1, y_1) had radius 20 m and mass $m_1 = m = 10^{-21}$ kg, Disc II (with forced degrees of freedom x_2, y_2) had radius 20 m and mass $m_2 = m$, Disc III (with unforced degrees of freedom x_3, y_3) had radius 80 m and mass $m_3 = m$, the "handle" discs at the ends of the string (with position parameterized by unforced degree of freedom y_4) had radius 40 m and mass $m_4 = 0.1m$, and the string had length 400 m. Disc II was initially positioned with a vertical offset of $0.1L$ from Disc I. The allowed region in system phase space is indicated in Table IV, with momentum components bounded by $|p_j(t)| \leq m_j(q_j^{\max} - q_j^{\min})/\epsilon$. All disc-disc collisions were perfectly elastic (with the exception of perfectly inelastic disc-handle collisions perpendicular to the string), and all disc-boundary collisions were perfectly inelastic. The vertical degree of freedom of Disc III (y_3) was subject to a drag force proportional to velocity. Causal entropic forces on Discs I and II were calculated independently from a common estimated global causal path distribution, giving neither disc a detailed "theory of mind" of the other.

The stochastic equations of motion for the evolution of path microstates (which were sampled to calculate the causal entropic forces) were

$$\dot{p}_{x_1}(t) = g_{x_1}(\mathbf{x}(t), t) = -p_{x_1}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_{x_1}(\lfloor t/\epsilon \rfloor \epsilon) + h_{x_1}(\mathbf{x}(t)) \tag{45}$$

$$\dot{p}_{y_1}(t) = g_{y_1}(\mathbf{x}(t), t) = -p_{y_1}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_{y_1}(\lfloor t/\epsilon \rfloor \epsilon) + h_{y_1}(\mathbf{x}(t)) \tag{46}$$

$$\dot{p}_{x_2}(t) = g_{x_2}(\mathbf{x}(t), t) = -p_{x_2}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_{x_2}(\lfloor t/\epsilon \rfloor \epsilon) + h_{x_2}(\mathbf{x}(t)) \tag{47}$$

$$\dot{p}_{y_2}(t) = g_{y_2}(\mathbf{x}(t), t) = -p_{y_2}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_{y_2}(\lfloor t/\epsilon \rfloor \epsilon) + h_{y_2}(\mathbf{x}(t)) \tag{48}$$

$$T(t) = \frac{2h_{y_4}(\mathbf{x}(t))m_3/m_4 - 2h_{y_3}(\mathbf{x}(t))}{2m_3/m_4 + 4} \tag{49}$$

$$\dot{p}_{x_3}(t) = h_{x_3}(\mathbf{x}(t)) \tag{50}$$

$$\dot{p}_{y_3}(t) = h_{y_3}(\mathbf{x}(t)) + 2T(t) - D\dot{q}_{y_3}(t) \tag{51}$$

$$\dot{p}_{y_4}(t) = h_{y_4}(\mathbf{x}(t)) - T(t) \tag{52}$$

$$\dot{q}_{x_1}(t) = p_{x_1}(t)/m \tag{53}$$

$$\dot{q}_{y_1}(t) = p_{y_1}(t)/m \tag{54}$$

$$\dot{q}_{x_2}(t) = p_{x_2}(t)/m \tag{55}$$

$$\dot{q}_{y_2}(t) = p_{y_2}(t)/m \tag{56}$$

$$\dot{q}_{x_3}(t) = p_{x_3}(t)/m \tag{57}$$

$$\dot{q}_{y_3}(t) = p_{y_3}(t)/m \tag{58}$$

$$\dot{q}_{y_4}(t) = p_{y_4}(t)/(0.1m), \tag{59}$$

and the deterministic equations of motion for the evolution of the causal macrostate (shown in Figure 4 and Supplemental Movie 4) when subjected to the combined causal entropic force and

expectation energetic force contributions were

$$\dot{p}_{x_1}(t) = \langle g_{x_1}(\mathbf{x}(t), t)\rangle + F_{x_1}(t) \tag{60}$$

$$= -p_{x_1}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + F_{x_1}(t) + h_{x_1}(\mathbf{x}(t)) \tag{61}$$

$$\dot{p}_{y_1}(t) = \langle g_{y_1}(\mathbf{x}(t), t)\rangle + F_{y_1}(t) \tag{62}$$

$$= -p_{y_1}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + F_{y_1}(t) + h_{y_1}(\mathbf{x}(t)) \tag{63}$$

$$\dot{p}_{x_2}(t) = \langle g_{x_2}(\mathbf{x}(t), t)\rangle + F_{x_2}(t) \tag{64}$$

$$= -p_{x_2}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + F_{x_2}(t) + h_{x_2}(\mathbf{x}(t)) \tag{65}$$

$$\dot{p}_{y_2}(t) = \langle g_{y_2}(\mathbf{x}(t), t)\rangle + F_{y_2}(t) \tag{66}$$

$$= -p_{y_2}(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + F_{y_2}(t) + h_{y_2}(\mathbf{x}(t)) \tag{67}$$

$$T(t) = \frac{2h_{y_4}(\mathbf{x}(t))m_3/m_4 - 2h_{y_3}(\mathbf{x}(t))}{2m_3/m_4 + 4} \tag{68}$$

$$\dot{p}_{x_3}(t) = h_{x_3}(\mathbf{x}(t)) \tag{69}$$

$$\dot{p}_{y_3}(t) = h_{y_3}(\mathbf{x}(t)) + 2T(t) - D\dot{q}_{y_3}(t) \tag{70}$$

$$\dot{p}_{y_4}(t) = h_{y_4}(\mathbf{x}(t)) - T(t) \tag{71}$$

$$\dot{q}_{x_1}(t) = p_{x_1}(t)/m \tag{72}$$

$$\dot{q}_{y_1}(t) = p_{y_1}(t)/m \tag{73}$$

$$\dot{q}_{x_2}(t) = p_{x_2}(t)/m \tag{74}$$

$$\dot{q}_{y_2}(t) = p_{y_2}(t)/m \tag{75}$$

$$\dot{q}_{x_3}(t) = p_{x_3}(t)/m \tag{76}$$

$$\dot{q}_{y_3}(t) = p_{y_3}(t)/m \tag{77}$$

$$\dot{q}_{y_4}(t) = p_{y_4}(t)/(0.1m), \tag{78}$$

where $g_j(\mathbf{x}(t), t)$ represent the energetic force components defined in the main text, $f_j(t)$ represent the random force components defined in the main text, $F_j(t)$ represent the causal entropic force components defined in the main text, $h_j(\mathbf{x}(t))$ represent any instantaneous disc-disc or disc-boundary collision forces, $D \equiv 0.1/\epsilon$ is the drag coefficient, and T is the string tension.

As control experiments – performed implicitly as part of the Monte Carlo path integral calculations – the model was stochastically evolved $1,000$ times from its initial configuration for duration τ. In only 39 out of $1,000$ runs (3.9%) was such random evolution able to pull Disc III to a position accessible from either of the compartments such that $q_{y_3}(\tau) \geq 0.3L$.

PATH INTEGRAL CALCULATION DETAILS

The causal entropic force path integral (11) can be calculated by a variety of techniques. In the case of the example simulations, we chose a Monte Carlo approach, as follows. Consider M stochastically sampled paths $\mathbf{x_i}(t)$ starting from the initial system state $\mathbf{x}(0)$, where the sample index is denoted by i and the respective initial effective fluctuating force components are denoted by $f_{ij}(0)$, such that $\Pr(\mathbf{x_i}(t)|\mathbf{x}(0)) > 0$. For large M, we can estimate the conditional path distribution $\Pr(\mathbf{x_i}(t)|\mathbf{x}(0))$ as being uniform over a local patch of path-space around each path $\mathbf{x_i}(t)$ with volume $\Omega_i \equiv [M\Pr(\mathbf{x_i}(t)|\mathbf{x}(0))]^{-1}$, such that probability is conserved: $\sum_i \Omega_i \Pr(\mathbf{x_i}(t)|\mathbf{x}(0)) = 1$. The path integral (11) can then be written as

$$F_j(\mathbf{X_0}, \tau) = -\frac{2T_c}{T_r} \int_{\mathbf{x}(t)} f_j(0) \Pr(\mathbf{x}(t)|\mathbf{x}(0)) \ln \Pr(\mathbf{x}(t)|\mathbf{x}(0))\, \mathcal{D}\mathbf{x}(t) \tag{79}$$

$$\approx -\frac{2T_c}{T_r} \left\langle \sum_i f_{ij}(0)(M\Omega_i)^{-1} \ln(M\Omega_i)^{-1}\Omega_i \right\rangle \tag{80}$$

$$= -\frac{2T_c}{T_r}\frac{1}{M} \left\langle \sum_i f_{ij}(0) \ln(M\Omega_i)^{-1} \right\rangle = \frac{2T_c}{T_r}\frac{1}{M} \left(\ln M \sum_i \left\langle f_{ij}(0) \right\rangle + \left\langle \sum_i f_{ij}(0) \ln \Omega_i \right\rangle \right) \tag{81}$$

$$= \frac{2T_c}{T_r}\frac{1}{M} \left\langle \sum_i f_{ij}(0) \ln \Omega_i \right\rangle = \frac{2T_c}{T_r}\frac{1}{M} \left(\sum_i \left\langle f_{ij}(0) \ln \Omega_i \right\rangle - \left\langle \ln \sum_{i'} \Omega_{i'} \right\rangle \sum_i \left\langle f_{ij}(0) \right\rangle \right) \tag{82}$$

$$= \frac{2T_c}{T_r}\frac{1}{M} \left\langle \sum_i f_{ij}(0) \ln \frac{\Omega_i}{\langle \sum_{i'} \Omega_{i'} \rangle} \right\rangle \approx \left\langle \frac{2T_c}{T_r}\frac{1}{M} \sum_i f_{ij}(0) \ln \frac{\Omega_i}{\sum_{i'} \Omega_{i'}} \right\rangle, \tag{83}$$

using the fact that $\langle f_{ij}(0) \rangle = 0$. Note that the final form is expressed in terms of patch volume fractions $\Omega_i / \sum_{i'} \Omega_{i'}$, which can be calculated by kernel density estimation applied to the trajectory set $\{\mathbf{x_i}(t_n)\}$ over all samples i.

For reference, pseudocode for performing the particle in a box simulation using this Monte Carlo approach is shown below. Our full causal entropic force simulation software will be made available for exploration at `http://www.causalentropy.org`.

Pseudocode for particle in a box example

```
function CALCULATE_CAUSAL_ENTROPIC_FORCE(cur_macrostate)
        /* --- Monte Carlo path sampling --- */
        sample_paths = EMPTY_PATH_LIST();
        for i = 1 to num_sample_paths do
                cur_path = EMPTY_PATH();
                cur_state = cur_macrostate;
                for n = 0 to num_time_steps do
                        cur_path[n] = cur_state;
                        cur_state = STEP_MICROSTATE(cur_state);
                sample_paths[i] = cur_path;
        /* --- Kernel density estimation of log volume fractions --- */
        log_volume_fracs = LOG_VOLUME_FRACTIONS(sample_paths);
        /* --- Sum force contributions --- */
        force = ZERO_FORCE_VECTOR();
        for i = 1 to num_sample_paths do
                force += sample_paths[i].initial_force * log_volume_fracs[i];
        return 2 * (T_c / T_r) * (1.0 / num_sample_paths) * force;

function PERFORM_CAUSAL_ENTROPIC_FORCING(init_macrostate)
        cur_macrostate = init_macrostate;
        while True do
                causal_entropic_force = CALCULATE_CAUSAL_ENTROPIC_FORCE(cur_macrostate);
                cur_macrostate = STEP_MACROSTATE(cur_macrostate,causal_entropic_force);

PERFORM_CAUSAL_ENTROPIC_FORCING({"q_x":L/10, "q_y":L/10, "p_x":0, "p_y":0});
```


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